Continuum Limit of Mean-Field Partition Functions

Deriving PDEs for partition functions by taking degrees of freedom to the continuum limit

1 Introduction

When analyzing a collection of one-dimensional harmonic oscillators, connected end-to-end, we often start with the equation of motion

$$\ddot{x}_j(t) = \omega_0^2 \left(x_{j+1}(t) + x_{j-1}(t) - 2x_j(t)\right). \tag{1}$$

Introducing a lattice spacing of a between adjacent oscillators and then taking this spacing to 0, we can show that Eq.(1) becomes the wave equation for a string. Such limits (termed "continuum limits") are used repeatedly throughout physics to derive higher dimensional partial differential equations from more local and lower dimensional ordinary differential equations.

Such transformations, typically applied to the dynamic degrees of freedom of systems, can also be applied to more abstract theoretical constructs. For some statistical physics systems, the partition function obeys a recursion relation reminiscent of equations like Eq.(1), i.e., equations that are the starting point for a continuum limit that results in a PDE. For example, a simple lattice model with the partition function $Z_N(\beta\lambda) = (1 + e^{-\beta\lambda})^N$, obeys the recursion relation

$$Z_N'(\beta\lambda) = N\left(Z_N(\beta\lambda) - Z_{N-1}(\beta\lambda)\right) \tag{2}$$

Formalizing recursion relations like Eq.(2), we should be able to find "continuum limits" for the associated partition functions, and such limits should lead to PDEs whose solutions mirror the solutions of the original system.

2 Main Result

If we have a partition function Z_N whose microstate degrees of freedom can be expressed in terms of $\sigma_i \in \{0, 1\}$, and whose energy $\mathcal{E}$ and denegerarcy of states ω are finite functions of the "mean-field" $\sum_i \sigma_i$, then the system partition function takes the form

$$Z_N(\beta) = \sum_{\boldsymbol{\sigma}} \omega\Big(\sum_i \sigma_i\Big) \exp\left[-\beta\mathcal{E}\Big(\sum_i \sigma_i\Big)\right]. \tag{3}$$

We define the total of microstates at infinite temperature as $\Omega(N) \equiv \sum_{\boldsymbol{\sigma}} \omega(\sum_i \sigma_i)$. By placing the partition functions on a lattice for different Ns, and then taking the lattice to the continuum limit, we find that the continuum analog of Z_N obeys the partial differential equation

$$\left[\frac{\partial}{\partial\beta} + \mathcal{E}\left(\frac{\partial}{\partial \ln\eta}\right)\right] \mathcal{Z}[\eta; \beta\mathcal{E}] = 0, \tag{4}$$

where the "initial condition" is given by $Z[\eta; 0] = \Omega(\eta)$. Upon solving Eq.(4) for various systems, we find that the result does not reproduce the exact functional forms of the partition functions for the original system, but it does reproduce important tranisition temperatures. For example, for the statistical physics of the symmetric group the original partition function and resulting continuum limit are

$$Z_N = \int_0^\infty dx\, e^{-x}\left(1 + (x-1)e^{-\beta\lambda}\right)^N \quad \xrightarrow{\text{continuum limit}} \quad Z(\eta) = \Gamma(\eta e^{-\beta\lambda} + 1). \tag{5}$$

Analyzing the latter result, we find that transition temperature for the latter $k_B T_c = \lambda/\ln(\eta/\varepsilon)$ (for $\varepsilon \approx 0.46$) matches the functional form of the $k_B T_c = \lambda/\ln N$ transition temperature of the former.